5.5

Radical! Because It's Cliché!

Properties of Rational Exponents

LEARNING GOALS

In this lesson, you will:

- Simplify expressions with negative exponents.
- Simplify expressions with rational exponents.
- Write negative powers as positive powers.
- Write rational powers using radicals.
- Find the *n*th root of a number.
- Write an expression in radical form.

KEY TERMS

- cube root
- index
- *n*th root
- radicand
- rational exponent

Where does the word "radical" come from? Why did mathematicians choose this word for the symbol $\sqrt{\ }$?

It turns out that the Latin word "radix" means "root." The word "radish"—a root vegetable—has the same origin as the word "radical." When you use radicals in mathematics, you are determining "roots" of numbers.

There are other words, too, that come from *radix*. The word "eradicate" is an example: to uproot or destroy.

PROBLEM 1 Negative Exponents

Three units of time that are smaller than seconds are milliseconds, microseconds, and nanoseconds.

1. Complete each statement to determine:

   a. the number of seconds there are in 100 milliseconds.

   $$100 \text{ milliseconds}\left(\frac{1 \text{ second}}{1000 \text{ milliseconds}}\right) = \frac{\square \text{ seconds}}{\square}$$

   $$= \_\_\_\_\_ \text{ second(s)}$$

   b. the number of seconds there are in 10 milliseconds.

   $$10 \text{ milliseconds}\left(\frac{1 \text{ second}}{1000 \text{ milliseconds}}\right) = \frac{\square \text{ seconds}}{\square}$$

   $$= \_\_\_\_\_ \text{ second(s)}$$

   c. the number of seconds there are in 1 millisecond.

   $$1 \text{ millisecond}\left(\frac{1 \text{ second}}{1000 \text{ milliseconds}}\right) = \frac{\square \text{ second}}{\square}$$

   $$= \_\_\_\_\_ \text{ second(s)}$$

5

Recall the Quotient Rule of Powers is:

$$\frac{a^m}{a^n} = a^{m-n}, \text{ when } a \neq 0$$

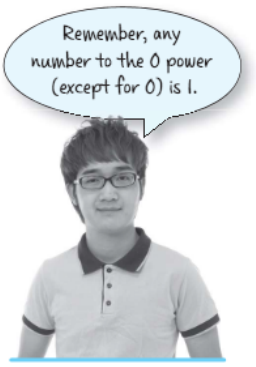
2. Show how you can use powers of 10 to write your answers from Question 1 as a quotient of powers and then as a single power.

   a. $\frac{100}{1000} =$ _____ $=$ _____

   b. $\frac{10}{1000} =$ _____ $=$ _____

   c. $\frac{1}{1000} =$ _____ $=$ _____

3. Explain how you can write a power with a negative exponent as a power with a positive exponent.



PROBLEM 2 Boat Stability

The weight of a boat depends on its beam width. An equation that relates a boat's weight to its beam width is $w = 64\left(\frac{b}{c}\right)^3$, where w is the boat's weight in pounds, b is the beam width in feet, and c is the capsize factor. When the capsize factor c is less than 2, the boat is less likely to capsize, or turn over.

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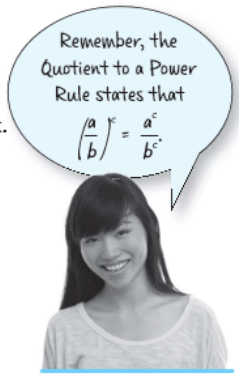
1. Suppose that a boat has a capsize factor of 3 and a beam width of 6 feet. What is the weight of the boat? Show your work.

2. Suppose that a boat has a capsize factor of 3 and a beam width of 9 feet. What is the weight of the boat? Show your work.

3. Explain how you calculated your answers to Questions 1 and 2.

4. Suppose that a boat has a capsize factor of 2 and weighs 1000 pounds.
   a. Write an equation that you can use to determine the beam width of the boat.

   b. Transform the equation you wrote to isolate the variable with its power on one side of the equation. Show your work.



   c. To determine the value of b, you need to know the number that when multiplied by itself 3 times is equal to 125. What is this number?

   d. What is the boat's beam width?

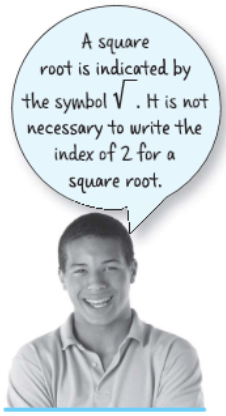
You can say that a number b is a **cube root** of a number a if $b^3 = a$.

So, 5 is a cube root of 125 because $5^3 = 125$.

A cube root is indicated by the symbol $\sqrt[3]{\ }$. The number 3 is called the **index** of the radical. So, $\sqrt[3]{125} = 5$.

5. Complete each statement.

a. $\sqrt[3]{8}$ – _____ **b.** $\sqrt[3]{-27}$ – _____



You can extend the idea of square roots and cube roots to solve equations. If n represents a positive number, then a number b is the ***n*th root** of a if $b^n = a$.

For example, 2 is the 4th root of 16 because $2^4 = 16$.

6. Complete each statement.

a. The number 3 is the 5th root of 243 because _____ = 243.

b. The number −2 is the cube root of −8 because $(-2)^3$ = _____.

c. The number 4 is the _____ root of 4096 because $4^6 = 4096$.

The nth root of a number a is designated as $\sqrt[n]{a}$, where n is the index of the radical and a is the **radicand**, which is the value within the radical.

7. Complete each statement. The first one has been done for you.

a. $\sqrt{100}$ = 10 because 10^2 = 100.

b. $\sqrt[3]{216}$ – _____ because _____^{3} = 216.

c. $\sqrt[4]{81}$ – _____ because _____^{4} = 81.

d. $\sqrt[5]{-32}$ – _____ because _____^{5} = −32.

8. Notice that a power can be positive or negative, depending on the base and the exponent.

   a. When the exponent of a power is an even number, and the base is a positive number, is the value of the power positive or negative? How do you know?

   b. When the exponent of a power is an even number, and the base is negative, is the value of the power positive or negative? How do you know?

   c. When the exponent of a power is an odd number, and the base is a positive number, is the value of the power positive or negative? How do you know?

   d. When the exponent of a power is an odd number, and the base is negative, is the value of the power positive or negative? How do you know?

5 PROBLEM 3 Boat Speed

The hull speed of a boat depends on the length of the hull at the waterline. An equation that relates the speed s in knots and the length, r in feet of the hull at the waterline is $s = 1.34\sqrt{r}$.

1. What is the hull speed of a boat that has a length of 16 feet at the waterline? Show your work.

2. Write an equation that you can use to determine the length of a boat at its waterline if the boat's hull speed is 6.7 knots.

3. Transform the equation you wrote to isolate the radical on one side of the equation. Show your work.

4. What is the length of the boat at the waterline? How do you know?

5. Complete each step.

$r = 25$	Given.
$r =$ _____	Write 25 as a power.
$r -$ _____	Substitute $\sqrt{r}$ for 5.

You can write the equation for the length of the boat at the waterline this way:

$$r^1 = (\sqrt{r})^2$$

You know that when you determine the power of a power, you multiply the exponents: $(a^b)^c = a^{bc}$.

Since $(\sqrt{r})^2 = r^1$, $\sqrt{r}$ must be equal to $r^{\frac{1}{2}}$, because $(r^{\frac{1}{2}})^2 = r^{(\frac{1}{2})(2)} = r^1$.

A **rational exponent** is an exponent that is a rational number. You can write each nth root using a rational exponent. If n is an integer greater than 1, then $\sqrt[n]{a} = a^{\frac{1}{n}}$.

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6. Write each radical as a power.

a. $\sqrt[3]{7}$ b. $\sqrt[5]{x}$ c. $\sqrt{y}$

7. Write each power as a radical.

a. $8^{\frac{1}{4}}$ b. $z^{\frac{1}{5}}$ c. $m^{\frac{1}{3}}$

8. Write the equation for the hull speed of a boat using a rational exponent.

Another equation can be used to determine the speed of a sailboat in knots. This equation is $s = \frac{a}{d^{\frac{2}{3}}}$, where a represents the sail area and d is the boat's displacement.

You can write the equation $s = \frac{a}{d^{\frac{2}{3}}}$ in radical form by using the properties you know about powers:

Given equation. $s = \frac{a}{d^{\frac{2}{3}}}$

Write the power as a product using a unit fraction. $s = \frac{a}{d^{\frac{1}{3}2}}$

Use the power of a power rule and the definition of a rational exponent to write the power as a radical. $s = \frac{a}{(\sqrt[3]{d})^2}$

9. Use the equation to calculate the speed s of a boat that has a sail area of 600 square feet and a displacement of 125 cubic feet. Show your work.

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10. Write each expression in radical form. Show your work and simplify your answer, if possible.

a. $4^{\frac{3}{2}}$

b. $5^{\frac{3}{4}}$

c. $x^{\frac{4}{5}}$

d. $y^{\frac{2}{3}}$

11. Write each expression in rational exponent form. Show your work and simplify your answer, if possible.

a. $(\sqrt[4]{2})^3$

b. $(\sqrt{5})^4$

c. $(\sqrt[5]{x})^5$

d. $(\sqrt[5]{y})^{10}$

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Be prepared to share your solutions and methods.